Theoretical Computer Science 14 (1981) 211–226
North-Holland Publishing Company

PATTERN DECOMPOSITION FOR TESSELLATION AUTOMATA

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Communicated by A. Salomaa
Received October 1979
Revised March 1980

Abstract. In this paper we obtain the following two results about one-dimensional tessellation automata. (1) There is a class of one-dimensional monogenic cellular automata with an arbitrary neighborhood size and arbitrary number of states where any pattern is always decomposed into atomic pieces leaving no residue. (2) For any positive integer q there exists a one-dimensional strongly connected tessellation automaton with q states even if we put the restriction that available parallel transformation must be bijective mappings on the set of finite configurations.

1. Introduction

It is demonstrated in the literature [1–3] that a decomposition phenomenon can be seen in a monogenic tessellation automata. That is, if one places any finite pattern into some monogenic tessellation space the pattern will decompose into atomic 'pieces' in a uniform way, and in some cases will leave a 'residue'. By monogenic, we mean that only one global map is allowed for the transformation that specifies the tessellation automaton at hand. The residue will remain situated in the same cells as the transformation is further applied repeatedly. However, if one alters the transformation momentarily, the residue dissolves into the atomic pieces as well. The first result we verify in this paper is that there is a class of one-dimensional monogenic cellular automata with an arbitrary number of states and with arbitrary neighborhood size where any pattern is always decomposed into atomic pieces, leaving no residue (Theorem 13 and 14). The form of the atomic pieces are determined by the state set and by the neighborhood index.

In the monogenic tessellation automaton, each cell works simultaneously according to the same transition function. So, once a configuration becomes uniform, the uniformity will remain in a tessellation space during successive applications of a global transformation. In some cases, we can construct a tessellation automaton with the property that certain configurations which are not uniform can be transformed into uniform configurations by successive applications of a global transformation. The firing squad synchronization problem can be considered to be a problem of constructing finite tessellation automata with the property. In the problem the initial configuration which is not uniform is the one which associates the same state with all cells except the general cell and the final uniform configuration is the one which associates the firing state with all cells. The problem is not trivial because we put the restriction that the firing states are not allowed to appear before the uniform configuration is obtained. The tessellation automata, where the decomposition phenomenon is observed, can be thought of as the one that has the property mentioned above. In the tessellation automata, we can initially take any finite configuration which is not uniform, and we can finally obtain a uniform configuration composed of atomic pieces. The configuration composed of atomic pieces is uniform in the sense that the finite part of the configuration is the repeated form of some kinds of atomic pieces.

The decomposition results above lead to a solution of the completeness problem [1] for polygenic tessellation automata, namely: For an arbitrary given tessellation automaton, are there patterns that can never be assumed by the array no matter what sequence of global transformations is applied to a certain canonical initial pattern? When such a pattern does not exist, the tessellation automaton is said to be complete. Many results concerned with the completeness problem have been obtained in the literature [1–4]. But they deal exclusively with the cases where any global transformations that can be defined in the tessellation automaton is available. The second result we obtain in this paper is that one-dimensional tessellation automata with arbitrary number of states are complete even if we put the restriction that available global transformations must be bijective mappings on the set of finite configurations (Theorem 18 and 19). The results are stated in terms of strong connectivity, which is equivalent notion to completeness.

2. Decomposition phenomenon for monogenic tessellation automata

In the main we employ the definition and notation given by Yamada and Amoroso [1].

The abbreviation TA is used to mean *Tessellation automaton.* By a *one-dimensional, q-state, scope-n* TA, we mean TA of the form

$$(\Sigma, Z, X, T),$$

where $\Sigma = \{0, 1, 2, \ldots, q-1\}$ is the set of states that can be assumed by any *cell* (finite-state machine) in the one-dimensional array. Z is the set of integers which is used to name the cells of the one-dimensional array. $X = (x_1, \ldots, x_n)$ is called the *neighborhood index* of the TA and is used to define the uniform interconnection pattern, where $x_1, \ldots, x_n \in Z$. In this paper, we assume $x_{i+1} = x_i + 1$, for $i = 1, \ldots, n-1$, and abbreviate $X = (x_1, \ldots, x_n)$ by $X_{(x_1, x_n)}$. A configuration, denoted by c, is defined as a mapping from Z to Σ and is used to denote the state of the TA as a whole. The image of i under c, $c(i)$, is the state assumed by cell i. C denotes the set of all the configurations with respect to Z and Σ. A mapping σ from Σ^n to Σ is called a *local map*. A *parallel map* is a mapping from C to C defined from local map σ and neighborhood index X as follows. That is, $\tau(c) = c'$ if and only if, for any cell $i \in Z$, $c'(i) = \sigma(c(i+x_1), \ldots, c(i+x_n))$. $0 \in \Sigma$ is called the *quiescent state*. A configuration c is called finite if and only if $c(i) = 0$ for all but finitely many cell i. In what follows, we are exclusively concerned with finite configurations. The set of all finite configurations is denoted by C_F. It is easy to see that C_F is closed for a given parallel map τ (i.e., for any $c \in C_F$, $\tau(c)$ is also in C_F) if and only if τ is defined from local map σ with $\sigma(0, \ldots, 0) = 0$. We define T to be a subset of T_F, where T_F denotes the set of all the finite configurations preserving parallel maps for the TA. In this paper, we are exclusively concerned with one-dimensional TA.

To begin with, we give an alternative mechanism to define mappings from C_F to C_F, which is convenient for discussions to follow. Throughout the paper, we assume $k \geq 2$ is an integer fixed arbitrarily. As is mentioned later, k specifies the form of the atomic pieces and is also related to the size of the neighborhood index.

Definition 1. Let i, j be integers such that $i \leq j$. (i, j) is an *interval* of $c \in C_F$ if (i) or (ii) is satisfied.

(i) $c(i-1) \neq c(i)$, $c(j) \neq c(j+1)$, and there exist $s \geq 1$, $t \geq 0$, $a \in \Sigma - \{0\}$ such that $c(i)c(i+1) \cdots c(j) = a^s 0^t$.

(ii) $j - i > k$, $c(i-1) \neq c(i)$, $c(r) = 0$ for any $r > j$, and there exists $a \in \Sigma - \{0\}$ such that $c(i)c(i+1) \cdots c(j) = a^{j-i-k} 0^{k+1}$.

Let c be an arbitrary finite configuration. Since the one-dimensional array of cells is uniquely partitioned into the intervals, we can alternatively define a mapping f from C_F to C_F by giving correspondence between $c(i) \cdots c(j)$ and $f(c)(i) \cdots f(c)(j)$, where (i, j) is an interval or union of two intervals of c. Through the following correspondence between strings we can define a mapping from C_F to C_F, which plays a central role throughout the paper. Let $a \in \Sigma - \{0\}$ be an arbitrary nonquiescent state.

$$\Big[\begin{smallmatrix} a0^j \\ a0^j \end{smallmatrix}, \qquad 0 \leq j \leq k-2, \tag{1}$$

$$\Big[\begin{smallmatrix} a0^{k+j-1} \\ a^k 0^j \end{smallmatrix}, \qquad j \geq 0, \tag{2}$$

$$\Big[\begin{smallmatrix} a^i 0^j \\ a^i 0^j \end{smallmatrix}, \qquad 2 \leq i \leq k-1,\ 0 \leq j \leq k, \tag{3}$$

$$\Big[\begin{smallmatrix} a^i 0^j \\ a0^{i-1} a0^{j-1} \end{smallmatrix}, \qquad 2 \leq i \leq k-1,\ k+1 \leq j, \tag{4}$$

$$\Big[\begin{smallmatrix} a^{k+i} 0^j \\ a0^{k-1} a^i 0^j \end{smallmatrix}, \qquad i \geq 0,\ j \geq 0, \tag{5}$$

$$\Big[\begin{smallmatrix} a0^{i-1} a^k 0^{j-k} \\ a^i 0^j \end{smallmatrix}, \qquad 2 \leq i \leq k-1,\ k+1 \leq j. \tag{6}$$

Let us denote by Σ^+ the set of all strings composed of symbols in Σ, excluding the null string. By $\Big[\begin{smallmatrix} \eta \\ \gamma \end{smallmatrix}$, where $\eta, \gamma \in \Sigma^+$, we mean η corresponds to γ. Through the correspondence given by (1)–(6), we define a mapping from C_F to C_F, which we denote by τ_1. That is, for any $c \in C_F$, $\tau_1(c)$ is determined as follows. Let $c' = \tau_1(c)$.

(i) Let (l, m) and $(m+1, r)$ be intervals of c. Put $\eta = c(l)c(l+1)\cdots c(r)$. If there exists $[^{\eta}_{\gamma}$ of the form (6), let $c'(l)c'(l+1)\cdots c'(r) = \gamma$. This process is made for all the adjacent intervals.

(ii) For any interval (l, m) that remains unspecified by (i), let $c'(l)c'(l+1)\cdots c'(m) = \gamma$, where $\eta = c(l)c(l+1)\cdots c(m)$ and $[^{\eta}_{\gamma}$ belongs to the correspondence. Note that $[^{\eta}_{\gamma}$ cannot be of the form (6).

(iii) For any cell l unspecified by (i) and (ii), let $c'(l) = 0$.

It is easy to see that τ_1 is well defined. Note that, when $k = 2$, the correspondence is given solely by (1), (2), and (5).

Example 1. If $c \in C_F$ is such that $c(l+i) = \alpha_i$, $0 \leq i \leq j$, for some integer l and positive integer j, and if $L_0(c) \geq l$ and $R_0(c) \leq l+j$, then c is represented by

$$\bar{0}\alpha_0\alpha_1\cdots\alpha_{l+j}\bar{0}.$$

If we are unconcerned with the symbols between α_s and α_t, then we write the configuration as

$$\bar{0}\alpha_0\alpha_1\cdots\alpha_s - \alpha_t\alpha_{t+1}\cdots\alpha_{l+j}\bar{0}.$$

Let $c \in C_F$ be the configuration such that

$$c = \bar{0}\,1\,0\,1\,1\,1\,1\,0\,2\,0\,2\,2\,2\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,\bar{0}.$$

Assume that the cell whose state is the leftmost 1 is 0. Let $k = 4$. Then intervals of c are (0, 1), (2, 6), (7, 8), (9, 16), and (17, 27). From the definition of τ_1 we can see that relative positions of c and $\tau_1(c)$ are

$$c = \bar{0}\,1\,0\,1\,1\,1\,1\,0\,2\,0\,2\,2\,2\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,\bar{0},$$

$$\tau_1(c) = \bar{0}\,1\,1\,0\,0\,0\,0\,0\,2\,0\,2\,0\,0\,2\,0\,0\,0\,0\,1\,0\,0\,0\,1\,1\,\bar{0}.$$

Note that a symbol for $\tau_1(c)$ is written directly below a symbol for c if these symbols represent the states of the same cell for c and $\tau_1(c)$.

Lemma 1. *There exists a parallel map whose restriction to C_F is τ_1. The neighborhood index to define the parallel map is $X_{(-k,2k-2)}$ when $k \geq 3$, and is $X_{(-2,0)}$ when $k = 2$. Furthermore, these neighborhood indices are minimum in size.*

The proof of Lemma 1 is tedious and is omitted here. It involves constructing local maps from Σ^{3k-1} to Σ for the case with $k \geq 3$, and from Σ^3 to Σ for the case with $k = 2$.

We now proceed to discuss how c can be determined from $\tau_1(c)$. Let $L_0(c)$ denote the leftmost cell that takes nonquiescent state. That is, $c(L_0(c)) \neq 0$ and $c(j) = 0$ for any $j < L_0(c)$. Similarly, let $R_0(c)$ denote the rightmost cell that takes nonquiescent state.

Definition 2. Let A be a nonempty subset of Σ^+. A sequence of pairs of integers $((h_1, t_1), \ldots, (h_s, t_s))$, $s \geqslant 0$, is a *framing of* $c \in C_F$ *with respect to* A if the following three conditions are satisfied:

(1) $h_1 = L_0(c)$,

(2) $t_i + 1 = h_{i+1}$ for $i = 1, \ldots, s-1$,

(3) $c(h_i)c(h_i+1) \cdots c(t_i) \in A$ for $i = 1, \ldots, s$.

Each pair of integers that composes a framing is called a *frame*. When (h_i, t_i) is a frame, h_i is called the *head* of frame (h_i, t_i).

Let

$$N_1 = \bigcup_{a \in \Sigma - \{0\}} \{a0^{k-1}a^i0^j \mid i \geqslant 0, j \geqslant 0\},$$

$$N_2 = \bigcup_{a \in \Sigma - \{0\}} \{a^i0^j \mid 1 \leqslant i \leqslant k, j \geqslant 0\},$$

$$N = N_1 \cup N_2.$$

Definition 3. A sequence of pairs of integers $((h_1, t_1), \ldots, (h_s, t_s))$ is a τ_1*-framing* of $c \in C_F$ if the following three conditions are satisfied:

(1) $((h_1, t_1), \ldots, (h_s, t_s))$ is a framing of c with respect to N,

(2) $t_s = R_0(c) + k + 1$,

(3) for any i, $1 \leqslant i \leqslant s-1$, and for any $j > t_i$, $c(h_i)c(h_i+1) \cdots c(j) \notin N$.

A frame of a τ_1-framing is called a τ_1*-frame*.

Example 2. Let $k = 4$ and let c be as in Example 1. Then τ_1-framing of $\tau_1(c)$ and the relative positions of c and $\tau_1(c)$ are

$$c = \bar{0}\,1\,0\,1\,1\,1\,1\,0\,2\,0\,2\,2\,2\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,\bar{0},$$

$$\tau_1(c) = \bar{0}\,\dot{1}\,1\,0\,0\,0\,0\,0\,\dot{2}\,0\,\dot{2}\,0\,0\,\dot{2}\,0\,0\,0\,0\,\dot{1}\,0\,0\,0\,1\,1\,\bar{0},$$

where $\cdot$ is placed above the heads of τ_1-frames.

The next lemma can be easily verified.

Lemma 2. *For any* $c \in C_F$, *there exists a unique* τ_1*-framing of* c.

From the example above, we may expect the following lemma to hold.

Lemma 3. *Let* τ_1*-framing of* c' *be* $((h_1, t_1), \ldots, (h_p, t_p))$. *For* c *defined as follows,* $\tau_1(c) = c'$ *holds.*

(i) *Let* i *be an arbitrary integer with* $1 \leqslant i < p$. *Put* $\gamma = c'(h_i) \cdots c'(t_{i+1})$. *If there exists* $[^{\eta}_{\gamma}$ *of the form* (4), *let* $c(h_i) \cdots c(t_{i+1}) = \eta$. *This process is made for all the adjacent* τ_1*-frames.*

(ii) *For any τ_1-frame (h, t) that remains unspecified in* (i), let $c(h) \cdots c(t) = \eta$, where $\gamma = c'(h) \cdots c'(t)$ *and* $[^{\eta}_{\gamma}$ *belongs to the correspondence.*
(iii) *For any cell i unspecified in* (i) and (ii), *let* $c(i) = 0$.

Proof. From (i), (ii), and (iii) in the lemma, c and c' can be written as

$$c = \bar{0}\eta_1\eta_2 \cdots \eta_j\bar{0}, \qquad c' = \bar{0}\gamma_1\gamma_2 \cdots \gamma_j\bar{0}, \tag{7}$$

where $[^{\eta_i}_{\gamma_i}$ belongs to the correspondence given by (1)–(6). Let i be an arbitrary integer such that $1 \leq i \leq j$. We shall show that the cell that contains the leftmost symbol of η_i is the head of an interval of c. This follows immediately when $i = 1$ or the rightmost symbol of η_{i-1} is 0. So assume $i \geq 2$, and $a_l \neq 0$, where $\eta_{i-1} = a_1 \cdots a_l$. Put $a_l = a$. It is easy to see that, if $a \neq 0$, $[^{\eta_{i-1}}_{\gamma_{i-1}}$ is of the form given by (1), (3) or (5). Hence $\eta_{i-1} = \gamma_{i-1} = a^m$, for some m, $1 \leq m \leq k-1$ or $\eta_{i-1} = a^m$, $\gamma_i = a0^{k-1}a^{m-k}$, for some m, $m \geq k$. Put $\eta_i = b_1 \cdots b_r$, $\gamma_i = b'_1 \cdots b'_r$. Then by (3) in Definition 3, $b'_1 \neq a$. Hence $b_1 \neq a$. Therefore, the cell that contains b_1 is the head of the interval of c. From the definition of τ_1, it suffices to show that, for any t, $1 \leq t < j$, $\eta_t\eta_{t+1}$ does not take the form $b0^{u-1}b^k0^v$, where $b \in \Sigma - \{0\}$, $2 \leq u \leq k-1$, and $v \geq 1$. Assume in contradiction that there exists t, $1 \leq t < j$ such that $\eta_t\eta_{t+1}$ take the form. Then $\gamma_t\gamma_{t+1} = b0^{u-1}b0^{k+v-1}$ in view of the forms given by (1) and (5). Then, by (i) of the lemma, it follows that $\eta_t\eta_{t+1} = b^u0^{k+v}$, which is a contradiction.

From Lemma 2 and 3, we have the following theorem.

Theorem 4. *Map τ_1 from C_F to C_F is bijective.*

Proof. From Lemma 2 and 3, τ_1 from C_F to C_F is surjective. Since surjectivity on C_F implies injectivity on C_F [6], the theorem follows.

Let c^R be defined as $c^R(i) = c(-i)$. We define τ_2 from C_F to C_F as $\tau_2(c) = c'$ if and only if $\tau_1(c^R) = c'^R$ for any $c, c' \in C_F$.

Example 3. Let $k = 4$ and $c_1 = c^R$, where c is as in Example 1. Then the relative positions of c_1 and $\tau_2(c_1)$ are

$$c_1 = \bar{0}\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,2\,2\,2\,0\,2\,0\,1\,1\,1\,1\,0\,1\,\bar{0},$$

$$\tau_2(c_1) = \bar{0}\,1\,1\,0\,0\,0\,1\,0\,0\,0\,0\,2\,0\,0\,2\,0\,2\,0\,0\,0\,0\,0\,1\,1\,\bar{0}.$$

For τ_2, analogues of Lemma 1 and Theorem 4 also hold. It might be noted here that parallel maps used in [1] and [3], where the decomposition phenomena with the residue are discussed, are τ_1 and τ_2 in the case with $k = 2$ and in the case with $k = 2$ and $q \geq 2$, respectively. It can be easily seen from the definitions, that τ_1 and τ_2 are interchangeable with each other by reversing the direction of the axis for the tessellation space. In order to obtain tessellation automata in which the decomposition phenomena without the residue are observed, it seems to be necessary that in addition to τ_1 we provide a parallel map which is not in the symmetric relation to τ_1. So we specify one more parallel map τ_3, which is defined by local map σ_3 from Σ^{2k} to Σ given as follows:

$$\sigma_3(x_{-k+1},\ldots,x_0,\ldots,x_k)=$$
$$=\begin{cases} 0, & \text{if there exists } i,\ 1\leqslant i\leqslant k-1, \text{ such that } x_{i-k}\neq x_{i-k+1},\ x_i\neq x_{i+1}, \text{ and} \\ & x_{i-k+1}=\cdots=x_i\neq 0, \\ x_i, & \text{if there exists } i,\ 1\leqslant i\leqslant k-1, \text{ such that } x_{i-k}\neq x_i,\ x_i\neq x_{i+1}, \text{ and} \\ & x_{i-k+1}=\cdots=x_{i-1}=0,\ x_i\neq 0, \\ x_0, & \text{otherwise.}\end{cases}$$

τ_3 is defined to be the parallel map specified by σ_3 and $X=(-k+1,\ldots,k)$. It is easy to see that, when $\tau_3(c)=c'$, symbols for c' are related to symbols for c as follows. For $a\in\Sigma-\{0\}$, if $c(i-k)\neq c(i-k+1)$, $c(i-k+1)\cdots c(i)=a^k$, and $c(i)\neq c(i+1)$, then $c'(i-k+1)\cdots c'(i)=0^{k-1}a$. If $c(i-k)\neq c(i)$, $c(i-k+1)\cdots c(i)=0^{k-1}a$, and $c(i)\neq c(i+1)$, then $c'(i-k+1)\cdots c'(i)=a^k$. With respect to the other cells, the symbols for c are the same as those for c'. Therefore, $\tau_3\tau_3$ becomes the identity mapping on C, where $\tau_3\tau_3$ means the composition of maps. Thus τ_3 is both injective and surjective on C. So we have the following proposition.

Proposition 5. *Map τ_3 from C to C is bijective.*

We note here that τ_3 in the case with $k=2$ and $q=2$ is one of the nontrivial bijective parallel maps on C for binary scope-four tessellation automaton discussed in [5]. In the rest of this section, we shall verify that the decomposition phenomena without the residue are observed in tessellation automata with parallel map defined by the composition of τ_3 and τ_1. Before we proceed to detailed discussion, we give an example below to illustrate how an arbitrary pattern of a configuration decomposes into atomic pieces in a specific way by repeated application of $\tau_3\tau_1$. In what follows, we denote $\tau_3\tau_1$ by π, where the composition of maps, $\tau_3\tau_1$, is defined as $\tau_1\tau_3(c)=\tau_1(\tau_3(c))$.

Example 4. Let $k=3$. For $c_0=\bar{0}\,2\,1\,0\,2\,0\,0\,2\,1\,1\,\bar{0}$, we denote as $c_{2t}=\pi^t(c_0)$, and $c_{2t+1}=\tau_3\pi^t(c_0)$, so that $c_{2t-1}=\tau_1^{-1}\pi^t(c_0)$, where τ_1^{-1} is the inverse mapping of τ_1. The sequence $\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$ is shown in Fig. 1, where for simplicity the left and right side $\bar{0}$'s are omitted. In the strings for c_{2t}, the underlined substrings are the atomic pieces and the central substrings without underline are the residues.

As illustrated in the example, for any $c\in C_F$ we can see the decomposition phenomena in both the sequence of predecessors of c and that of successors of c

c_{-6} 2 2 2 1 1 1 0 0 0 2 2 2 0 0 0 0 2 0 0 0 0 1 0 0 0 1

2 0 0 1 0 0 0 0 0 2 0 0 2 2 2 0 0 1 1 1 0 1 1 1

c_{-5} 2 2 2 1 1 1 0 0 0 2 2 2 2 0 0 0 0 1 0 0 0 1

2 0 0 1 0 0 0 2 2 2 2 0 0 1 1 1 0 1 1 1

c_{-4} 2 2 2 1 1 1 0 2 0 0 2 0 0 1 0 0 0 1

2 0 0 1 0 2 0 0 2 1 1 1 0 1 1 1

c_{-2} 2 2 2 1 0 2 2 2 2 1 0 0 0 1

2 1 0 2 2 2 2 1 0 1 1 1

c_0 2 1 0 2 0 0 2 1 1

2 2 2 1 0 2 0 0 2 1 1

c_2 2 0 0 1 0 2 2 2 2 1 0 1

2 2 2 1 1 1 0 2 2 2 2 1 0 1

c_4 2 0 0 1 0 0 0 2 0 0 2 1 0 1 1 1

2 2 2 1 1 1 0 2 2 2 0 0 2 1 0 0 0 1

c_6 2 0 0 1 0 0 0 2 0 0 0 0 2 1 1 1 0 1 1 1

2 2 2 1 1 1 0 2 2 2 0 0 2 2 2 0 0 1 0 0 0 1

c_8 2 0 0 1 0 0 0 2 0 0 0 0 2 0 0 0 0 1 1 1 0 1 1 1

Fig. 1

under π. In what follows, we shall mainly discuss about the phenomena observed in the latter sequence, i.e., $c, \pi(c), \pi^2(c), \ldots$

Let us define as

$$S_1 = \{a_1 0^{k-1} a_2 0^{k-1} \cdots a_i 0^{k-1} 0^m \mid a_1, \ldots, a_i \in \Sigma - \{0\}, m \geq 1, i \geq 1,$$
$$\forall j (1 \leq j \leq i-1), a_j \neq a_{j+1}\},$$
$$S_2 = \{0^m a_i^k a_{i-1}^k \cdots a_1^k \mid a_1, \ldots, a_i \in \Sigma - \{0\}, m \geq 1, i \geq 1,$$
$$\forall j (1 \leq j \leq i-1), a_j \neq a_{j+1}\}.$$

In what follows, strings in S_1 or S_2 are thought of as the atomic pieces. Let c be an arbitrary finite configuration. Configuration c can be represented as

$$\bar{0} \eta_1 \cdots \eta_l x_1 \cdots x_i \zeta_r \cdots \zeta_1 \bar{0}, \tag{8}$$

where $\eta_1, \ldots, \eta_l \in S_1$, $\zeta_1, \ldots, \zeta_r \in S_2$, $x_1, \ldots, x_i \in \Sigma$, $l \geq 0$, $r \geq 0$, and $i \geq 0$. It is assumed that substring $x_1 \cdots x_i$ is chosen so that the length of it becomes as short as possible, where by the length of string $x_1 \cdots x_i$ we mean the integer i. That is, when $i \geq 1$, any string in S_1 cannot be a prefix of $x_1 \cdots x_i$, and any string in S_2 cannot be a prefix of $x_1 \cdots x_i$. Furthermore, when $i \geq 1$, $x_1 \neq 0$ and $x_i \neq 0$. Clearly this representation is unique when $i \geq 1$. Without loss of generality, the cells for $x_1, \ldots, x_i$ in the representation are assumed to be $1, \ldots, i$, respectively. When $i \geq 1$, the forward residue of c, denoted by $\mathrm{Res}_{\mathrm{F}}(c)$, is defined to be the configuration as

$$\mathrm{Res}_{\mathrm{F}}(c)(j) = \begin{cases} c(j), & \text{if } 1 \leq j \leq i, \\ 0, & \text{otherwise.} \end{cases}$$

When $i = 0$, $\mathrm{Res}_{\mathrm{F}}(c)$ is defined to be $\bar{0}$, where $\bar{0}$ denote the quiescent configuration. The quiescent configuration $\bar{0}$ is defined as $\bar{0}(j) = 0$ for any $j \in Z$. We may think of $\mathrm{Res}_{\mathrm{F}}(c)$ as the undissolved part of c. Let $x_1 \cdots x_p$ be the longest length substring that is a prefix of both $x_1 \cdots x_i$ and a string in S_1. Similarly, let $x_s \cdots x_i$ be the longest length substring that is a suffix of both $x_1 \cdots x_i$ and a string in S_2. Clearly both $x_1 \cdots x_p$ and $x_s \cdots x_i$ are uniquely determined. Note that, since $x_1 \cdots x_i$ is chosen as short as possible, $x_1 \cdots x_p$ is a proper prefix of the string in S_1 and $x_s \cdots x_i$ is a proper suffix of the string in S_2. For a configuration c which has the representation given by (8) with $i \geq 1$, we define $L_1(c)$ and $R_1(c)$ as

$$L_1(c) = 1 + (p - 1), \qquad R_1(c) = 1 - (s - 1).$$

That is, $L_1(c)$ and $R_1(c)$ are the cells that contain x_p and x_s, respectively. Clearly, $L_1(c) \leq R_1(c)$.

Let c be an arbitrary finite configuration. It will be established through several lemmas that in the sequence $c, \pi(c), \pi^2(c), \ldots$, strings in $S_1(S_2)$ on the left (right) side move left (right) and that strings in S_1 or S_2 do not influence how the symbols in the undissolved part are transformed by the parallel map. Furthermore, the length of the undissolved part, $lg(\mathrm{Res}_{\mathrm{F}}(\pi^i(c))$, decreases, eventually going to zero. When $lg(\mathrm{Res}_{\mathrm{F}}(\pi^i(c)))$ becomes zero, we consider the decomposition is completed. By the length of $c \in C_{\mathrm{F}}$, $lg(c)$, we mean the nonnegative integer $R_0(c) - L_0(c) + 1$.

From the definitions of τ_1 and τ_3, the next lemma can be easily verified.

Lemma 6. *For any $c \in C_{\mathrm{F}}$,*

$$\mathrm{Res}_{\mathrm{F}}(\pi(c)) = \pi(\mathrm{Res}_{\mathrm{F}}(c)).$$

Lemma 7. *Let $c \in C_{\mathrm{F}}$ be of the form $- a_2 a_1^{k(m+1)}\bar{0}$, where $a_1 \in \Sigma - \{0\}$, $a_1 \neq a_2$, and $m \geq 1$. Then $\pi^m(c)$ is of the form $- 0^{k-1} a_1^k \bar{0}$, and $R_0(c) = R_0(\pi^m(c))$.*

Proof. It is easily seen that for any i, $1 \leq i \leq m$, the relative positions of $\pi^i(c)$, $\tau_3\pi^i(c)$, and $\tau_1\tau_3\pi^i(c)$ are

$$\begin{array}{lll} & & \overbrace{\hphantom{a_1 0^{k-1} a_1 \cdots a_1\bar{0}}}^{(m+1-i)k \text{ times}} \\ \pi^i(c) & -a_1 0^{k-1} & a_1 \cdots \qquad\quad a_1\bar{0}, \\ \tau_3\pi^i(c) & -a_1 0^{k-1} & a_1 a_1^{k-1} a_1 \cdots a_1 \bar{0}, \\ \tau_1\tau_3\pi^i(c) & \text{———} & a_1 0^{k-1} a_1 \cdots a_1 \bar{0}. \end{array}$$

Thus we have the lemma.

Lemma 8. *For $c \in C_F$, let $c' = \pi(c)$. If $L_0(c') < L_0(c)$, then c' is of the form $\bar{0}a_1 0^{k-1} a_2 -$, where $a_1 \in \Sigma - \{0\}$, $a_1 \neq a_2$. Furthermore, $L_1(c') > L_0(c)$.*

Proof. Since $L_0(c') = L_0(\tau_3(c))$, $L_0(c') < L_0(c)$ implies $L_0(\tau_3(c)) < L_0(c)$. So from the definitions of τ_3 and τ_1, the relative positions of c, $\tau_3(c)$, and $\tau_1\tau_3(c)$ are

$$\begin{array}{rl} c & \bar{0}a_1 a_2' \text{———}, \\ \tau_3(c) & \bar{0}a_1 a_1^{k-1} a_2 \text{———}, \\ c' = \tau_1\tau_3(c) & \bar{0}a_1 0^{k-1} a_2 \text{———}, \end{array}$$

where $a_1 \in \Sigma - \{0\}$, $a_1 \neq a_2'$, $a_1 \neq a_2$. Thus, we have $L_1(c') \geq L_0(c') + k = L_0(\tau_3(c)) + k = (k-1) + L_0(c) + k > L_0(c)$.

Lemma 9. *For $c \in C_F$, let $c' = \pi(c)$. If $R_0(c') > R_0(c)$, then there exists an integer m such that $\pi^m(c)$ is of the form $-a_2 a_1^k \bar{0}$, where $a_1 \in \Sigma - \{0\}$, $a_1 \neq a_2$. Furthermore, $R_1(\pi^m(c)) < R_0(c)$.*

Proof. Since $R_0(c') > R_0(c) = R_0(\tau_3(c))$, the rightmost interval of $\tau_3(c)$ is transformed through the forms given by (2) or (4).

Case 1. The rightmost interval of $\tau_3(c)$ is transformed by (2). In this case, the relative positions of c, $\tau_3(c)$, and $\tau_1\tau_3(c)$ are

$$\begin{array}{rl} c & \text{———} a_1\bar{0}, \\ \tau_3(c) & \text{———} a_1\bar{0}, \\ c' = \tau_1\tau_3(c) & \text{———} a_1 a_1^{k-1}\bar{0}. \end{array}$$

Let $r = R_0(c)$. It is easily seen that, if $c'(r-1) = a_1$, then the interval of $\tau_3(c)$ to the left of the rightmost one is transformed by form (2). Repeating the argument, we conclude that c' is of the form —— $a_2 a_1^{kp}\bar{0}$, where $a_1 \neq a_2$. Thus, letting $m = p$, the lemma is established from Lemma 7.

Case 2. The rightmost interval of $\tau_3(c)$ is transformed by (4). The relative positions of the sequence of successors of c for τ_3 and τ_1 are

$$\begin{array}{rl} c & \text{————} \quad a_1\bar{0}, \\ \tau_3(c) & \text{————} \quad a_1\bar{0}, \\ \tau_1\tau_3(c) & \text{———} a_1 0^{s-1} 0 \; a_1\bar{0}, \\ \tau_3\tau_1\tau_3(c) & \text{———} a_1 0^{s-1} 0 \; a_1\bar{0}, \\ \tau_1\tau_3\tau_1\tau_3(c) & \text{———} 0^{s-1} 0 \; a_1 a_1^{k-1}\bar{0}, \end{array}$$

where $1 \leq s \leq k-2$. Thus lemma holds for $m = 2$.

Lemma 10. *Let $c \in C_F$ be of the form $\bar{0}a_1 0^{k-1} a_2 0^{k-1} \cdots a_s 0^{k-1} a_{s+1}$———, and let $L_1(c)$ contain a_{s+1}, where $a_1, \ldots, a_{s+1} \in \Sigma - \{0\}$, $s \geq 1$, and for any i, $1 \leq i \leq s$, $a_i \neq a_{i+1}$. Then $\pi(c)$ is of the form*

$$\bar{0}a_1 0^{k-1} a_2 0^{k-1} \cdots a_s 0^{k-1} 0^{k-1} \text{———},$$

or

$$\bar{0}a_1 0^{k-1} a_2 0^{k-1} \cdots a_s 0^{k-1} a_{s+1} 0^{k-1} a_{s+2} \text{——},$$

where $a_{s+1} \neq a_{s+2}$. Furthermore, $L_1(\pi(c)) > L_1(c)$, or $L_0(\mathrm{Res}_F(\pi(c))) \geq L_1(c)$.

Proof. Let $L_1(c) = j$. If $c(j) \neq c(j+1)$, then the relative positions of c, $\tau_3(c)$, and $\tau_1\tau_3(c)$ are

$$\begin{array}{ll} c & \bar{0}\, a_1 0^{k-1}\, a_2 0^{k-1} \cdots a_s 0^{k-1}\, a_{s+1} \text{———}, \\ \tau_3(c) & \bar{0} a_1^{k-1} a_1 a_2^{k-1} a_2 \quad \cdots a_s a_{s+1}^{k-1} a_{s+1} \text{———}, \\ \tau_1\tau_3(c) & \bar{0} a_1 0^{k-1} a_2 0^{k-1} \quad \cdots 0\; a_{s+1}\; 0^{k-1} \text{———}. \end{array}$$

It is easy to see that $\tau_3(c)(j+1) \neq a_{s+1}$ so that we have $\tau_1\tau_3(c)(j+1) \neq a_{s+1}$. In this case, $L_1(\pi(c)) \geq j+1 > j = L_1(c)$. On the other hand, if $c(j) = c(j+1)$, then the relative positions are

$$\begin{array}{ll} c & \bar{0}\, a_1 0^{k-1}\, a_2 0^{k\infty} \cdots a_s 0^{k-1}\, a_{s+1} \text{———}, \\ \tau_3(c) & \bar{0} a_1^{k-1} a_1 a_2^{k-1} a_2 \quad \cdots a_s\, 0^{k-1}\, a'_{s+1} \text{———}, \\ \tau_1\tau_3(c) & \bar{0} a_1 0^{k-1} a_2 0^{k-1} \quad \cdots 0\; 0^{k-1} a'_{s+1} \text{———}, \end{array}$$

where $a'_{s+1} = a_{s+1}$ or $a'_{s+1} = 0$. In this case, $L_0(\mathrm{Res}_F(\pi(c))) \geq j = L_1(c)$.

Lemma 11. *Let $c \in C_F$ be of the form $-a_{s+1} a_s^k \cdots a_1^k \bar{0}$, and let $R_1(c)$ contain a_{s+1}, where $a_1, \ldots, a_{s+1} \in \Sigma - \{0\}$, $s \geq 1$, and for any i, $1 \leq i \leq s$, $a_i \neq a_{i+1}$. Then there exists an integer m such that $\pi^m(c)$ is of the*

$$\text{————}0^{k-1} a_s^k \cdots a_1^k \bar{0},$$

or

$$\text{———}a_{s+2} a_{s+1}^k a_s^k \cdots a_1^k \bar{0},$$

where $a_{s+1} \neq a_{s+2}$. Furthermore, $R_1(\pi^m(c)) < R_1(c)$, or $R_0(\mathrm{Res}_F(\pi^m(c))) \leq R_1(c)$.

Proof. Let $R_1(c) = j$.

Case 1. Integer j is the head of an interval of $\tau_3(c)$. In this case, $\pi(c)$ must be of the form ——$a_{s+2} a_{s+1}^{kp} a_s^k \cdots a_1^k \bar{0}$ by the same argument as the one in the proof of Lemma 9, where $a_{s+1} \neq a_{s+2}$. Then by a similar argument to the proof of Lemma 7, setting $m = p$, $\pi^m(c)$ is of the form ——$0^{k-1} a_{s+1}^k 0^{(k-1)(m-1)} a_s^k \cdots a_1^k \bar{0}$. Clearly $R_1(\pi^m(c)) < j = R_1(c)$.

Case 2. Integer j is not the head of any interval of $\tau_3(c)$. The relative positions of c, $\tau_3(c)$, and $\tau_1\tau_3(c)$ are

$$\begin{array}{ll} c & \text{———} a_{s+1}a_s^{k-1}a_s\cdots a_1^{k-1}a_1\bar{0}, \\ \tau_3(c) & \text{———} a_{s+1}0^{k-1}a_s\cdots 0^{k-1}a_1\bar{0}, \\ \tau_1\tau_3(c) & \text{———} \quad 0^{k-1}a_sa_s^{k-1}\cdots a_1a_1^{k-1}\bar{0}. \end{array}$$

In this case, we have $R_0(\mathrm{Res}_F(\pi(c)))\leqslant R_1(c)$.

Lemma 8, together with Lemma 6, says that, if the undissolved part of a configuration c grows to the left as π is applied, it takes the form $\bar{0}a_10^{k-1}a_2$ —. Hence by repeated application of Lemma 10 we can conclude that, once the form is constructed, in the sequence c, $\pi(c)$, $\pi^2(c)$, ... it grows to the right to obtain a string in S_1. We also have a similar statement from Lemma 9 and 11. More formally we have the next lemma.

Lemma 12. *For any $c\in C_F$ with $\mathrm{Res}_F(c)\neq\bar{0}$, there exists a nonnegative integer m such that $lg(\mathrm{Res}_F(c)>lg(\mathrm{Res}_F(\pi^m(c)))$.*

Proof. If there exists a nonnegative integer t such that $L_0(\mathrm{Res}_F(\pi^t(c)))>L_0(\mathrm{Res}_F(\pi^{t+1}(c)))$ or $R_0(\mathrm{Res}_F(\pi^t(c)))<R_0(\mathrm{Res}_F(\pi^{t-1}(c)))$, then from Lemma 6, 8 and 9, and by repeated application of Lemma 10 and 11, there exists a nonnegative integer m such that $lg(\mathrm{Res}_F(c))>lg(\mathrm{Res}_F(\pi^m(c)))$. On the other hand, if there exists t such that $L_0(\mathrm{Res}_F(\pi^t(c)))<L_0(\mathrm{Res}_F(\pi^{t+1}(c)))$ or $R_0(\mathrm{Res}_F(\pi^t(c)))>R_0(\mathrm{Res}_F(\pi^{t+1}(c)))$, then the lemma holds clearly. So we suppose in contradiction that, for any positive integer t, $L_0(\mathrm{Res}_F(c))=L_0(\mathrm{Res}_F(\pi^t(c)))$ and $R_0(\mathrm{Res}_F(c))=R_0(\mathrm{Res}_F(\pi^t(c)))$. From the latter equation, it is easy to see that, for any positive integer t, the rightmost interval of $\tau_3(\mathrm{Res}_F(\pi^t(c)))$ is transformed through the form given by (5) with $i\geqslant 1$, where i is as in (5). Therefore, for any $t>0$, there exists $i\geqslant 1$ such that $\tau_3(\mathrm{Res}_F(\pi^t(c)))$ is of the form

$$\text{———} a_2a_1^{k+i}\bar{0}, \tag{9}$$

where $a_1\in\Sigma-\{0\}$ and $a_1\neq a_2$. Then $\tau_1\tau_3(\mathrm{Res}_F(\pi^t(c)))=\mathrm{Res}_F(\pi^{t+1}(c))$ is of the form —— $a_2a_10^{k-1}a_1^i\bar{0}$. Hence, $\tau_3(\mathrm{Res}_F(\pi^{t+j}(c)))$ can not have the form given by (9), where j is the greatest integer not greater than i/k. This is a contradiction.

As an immediate consequence of Lemma 12, we have the next theorem.

Theorem 13. *For any $c\in C_F$, there exists a nonnegative integer m such that* $\mathrm{Res}_F(\pi^m(c))=\bar{0}$.

Note that, if $\mathrm{Res}_F(\pi^m(c))=\bar{0}$, then $\pi^m(c)$ takes the form $\bar{0}\eta_1\cdots\eta_l\zeta_r\cdots\zeta_1\bar{0}$, where $l\geqslant 0$, $r\geqslant 0$, $\eta_1,\ldots,\eta_l\in S_1$, and $\zeta_1,\ldots,\zeta_r\in S_2$. That is, the decomposition is completed in $\pi^m(c)$. We have arbitrarily chosen integer k which determines the forms of strings in S_1 and S_2. From Lemma 1 and the definition of σ_3, if $k \geqslant 3$, then it requires the neighborhoods of size $3k-1$ and size $2k$ to realize τ_1 and τ_3, respectively. So the neighborhood of size $5k-2$ is necessary to realize π because π is the composition of τ_1 and τ_3

We can establish similar results corresponding to the lemmas and the theorem, obtained so far, for the sequence of predecessors of any $c \in C_F$, i.e., c, $\pi^{-1}(c)$, $\pi^{-2}(c), \ldots$. Here we only present the result for it. Let us recall that c^R is defined as $c^R(i) = c(-i)$. The backward residue of c, denoted by $\mathrm{Res_B}(c)$, is defined as

$$\mathrm{Res_B}(c) = (\mathrm{Res_F}(c^R))^R.$$

Corresponding to Theorem 14, we can verify the next theorem.

Theorem 14. *For any $c \in C_F$, there exists a nonnegative integer m such that $\mathrm{Res_B}(\pi^{-m}(c)) = \bar{0}$.*

Before ending this section, we present the following statement without proof. That is, (4) and (6) can be replaced by (4)′ and (6)′ below, respectively, to obtain Theorem 13 and 14.

$$\left[{}^{a^i0^j}_{a0^{i-1}a^k0^{j-k}}, \quad 2 \leqslant i \leqslant k-1,\ k+1 \leqslant j, \qquad (4)'\right.$$

$$\left[{}^{a0^{i-1}a0^{j-1}}_{a^i0^j}, \quad 2 \leqslant i \leqslant k-1,\ k+1 \leqslant j. \qquad (6)'\right.$$

3. Strong connectivity for tessellation automata with bijective parallel maps

A tessellation automaton with set T of parallel maps is defined to be strongly connected if for any c, c' with $c \neq \bar{0}$, $c' \neq \bar{0}$, there exists $\xi \in T^*$ such that $\xi(c) = c'$, where $T^* = T^+ \cup \{\lambda\}$. The null string is denoted by λ. In this section, we show that TA whose parallel maps are bijective on C_F is strongly connected.

Let τ_R be defined as $\tau_R(c) = c'$ if and only if $c(i) = c'(i+1)$ for any $i \in Z$. Similarly, τ_L is defined as $\tau_L(c) = c'$ if and only if $c(i) = c'(i-1)$ for any $i \in Z$. The next two lemmas were verified in [2].

Lemma 15. *Let $k = 2$ and let $c_1, c_2 \in C_F$. If there exists $\tau \in \{\tau_1, \tau_2\}$ such that $\tau(c_1) = c_2$, then there exists $\xi \in \{\tau_1, \tau_2, \tau_L, \tau_R\}^+$ such that $\xi(c_2) = c_1$.*

Although Lemma 15 holds without any restriction on k, we will not use this result here.

The next lemma can be easily verified.

Lemma 16. *Let $\alpha_1, \alpha_{-1} \in \Sigma^+$ be such that $lg(\alpha_1) = lg(\alpha_{-1}) = l$. Assume that, if there exist $a, b \in \{1, -1\}$ and $x_1 \cdots x_s \in \Sigma^+$ such that $x_1 \cdots x_l = \alpha_a$ and $x_{s-l+1} \cdots x_s = \alpha_b$,* *then there exists $y_1 \cdots y_s \in \Sigma^+$ such that $y_1 \cdots y_l = \alpha_{-a}$ and $y_{s-l+1} \cdots y_s = \alpha_{-b}$. Then the next equation defines a mapping τ on C.*

$$\tau(c)(i) \cdots \tau(c)(i+l-1) = \begin{cases} \alpha_{-a} & \text{if } c(i) \cdots c(i+l-1) = \alpha_a, \\ c(i) \cdots c(i+l-1), & \text{otherwise.} \end{cases}$$

Furthermore, the following three statements hold:

(i) $\tau^{-1} = \tau$. *That is, the inverse of τ is τ,*

(ii) $\tau: C \to C$ *is a bijection,*

(iii) *there exists a local map: $\Sigma^{2l-1} \to \Sigma$ that, together with neighborhood index $X_{(-l+1,l-1)}$, defines τ.*

Note that, when $s \geq 2l$, the condition of the lemma is satisfied trivially. In what follows, let us denote the mapping defined in Lemma 16 by $[{\alpha_1 \atop \alpha_{-1}}]$.

The next lemma is established in [6].

Lemma 17. *If a parallel map $\tau: C \to C$ is injective, then $\tau: C_F \to C_F$ is surjective, where by $\tau: C_F \to C_F$ we mean the mapping obtained from $\tau: C \to C$ by restricting both the domain C and the range C to C_F.*

We denote by T_{FB} the set of all the surjective parallel maps from C_F to C_F that can be defined from the neighborhood index at hand. Since it has been verified in [6] that surjectivity of $\tau: C_F \to C_F$ implies injectivity of it, any mapping in T_{FB} is a bijective mapping from C_F to C_F. Let us denote the number of elements in Σ by $|\Sigma|$.

Theorem 18. *If $|\Sigma| = 2$, then* TA $(\Sigma, Z, X_{(-7,7)}, T_{FB})$ *is strongly connected.*

Proof. Let τ_1, τ_2, τ_3 be the parallel maps with $k=2$. Let T_1 be the set of the parallel maps that can be defined in the way described in Lemma 16 for any α_1 and α_{-1} in the case where the neighborhood index is $X_{(-7,7)}$. We put $T_2 = T_1 \cup \{\tau_1, \tau_2, \tau_3\}$. From Theorem 4, Lemma 1, 16 and from the definition of τ_2 and τ_3, we have $T_2 \subseteq T_{FB}$. Let c be an arbitrary finite configuration other than $\bar{0}$. It suffices to show that there exists $\xi \in T_{FB}^*$ such that $\xi(c_p) = c$, where c_p is defined as $c_p(0) = 1$ and $c_p(i) = 0$ for all $i \neq 0$. Because, since it follows from Lemma 15, (i) of Lemma 16, and from $\tau_3^{-1} = \tau_3$ that for any $\tau \in T_2$ there exists $\beta \in T_2^+$ such that $\tau^{-1} = \beta$, this implies that for any c' other than $\bar{0}$ there exists $\xi' \in T_{FB}^*$ such that $\xi'(c') = c_p$, so that $\xi\xi'(c') = \xi(c_p) = c$. By the way, from Theorem 13 there exist c_1 and $\xi_1 \in \{\tau_1, \tau_3\}^*$ such that $\mathrm{Res}_F(c_1) = \bar{0}$ and $c_1 = \xi_1(c)$. So c_1 is of the form

$$\bar{0}\eta_1 \cdots \eta_l \zeta_r \cdots \zeta_1 \bar{0}, \tag{10}$$

where $l \geq 0$, $r \geq 0$, $\eta_1, \ldots, \eta_l \in S_1$ and $\zeta_1, \ldots, \zeta_r \in S_2$. In the case where $\Sigma = \{0, 1\}$ and $k = 2$, S_1 and S_2 are written as

$$S_1 = \{100^m \mid m \geq 1\}, \qquad S_2 = \{0^m 11 \mid m \geq 1\}.$$

Note that, since $c \neq \bar{0}$, we can exclude the case where $l = 0$ and $r = 0$. For ξ_1 takes $\bar{0}$ to $\bar{0}$ and ξ_1 is injective map from C_F to C_F. We now show that we can assume $l \neq 0$ and $r \neq 0$ without loss of generality. Let $\tau_4 = \left[\begin{smallmatrix} 0&1&1&0&0&0 \\ 0&1&1&0&1&0 \end{smallmatrix}\right]$. In the case where $l = 0$ and $r \neq 0$, we put $c_2 = \tau_1\tau_4(c_1)$. It is easy to see that c_2 is of the form $\bar{0}100 - 011\bar{0}$. Therefore, from Lemma 6 and Theorem 13, there exists $\xi_2 \in \{\tau_1, \tau_3\}^*$ such that $\xi_2(c_2)$ is of the form (10) with $l \neq 0$ and $r \neq 0$. This follows since the substring 100 on the leftmost side of c_2 and the substring 011 on the rightmost side of it move to the left and to the right, respectively, as $\tau_1\tau_3$ is applied to c_2 to obtain $\xi_2(c_2)$. In the case where $l \neq 0$ and $r = 0$, let $\tau_5 = \left[\begin{smallmatrix} 0&1&0&0&0&0 \\ 0&1&0&1&1&0 \end{smallmatrix}\right]$. And put $c_2 = \tau_1\tau_2\tau_5(c_1)$. We can easily see that there exists $\xi_2' \in \{\tau_1, \tau_3\}^*$ such that $\xi_2'(c_2)$ takes the form given by (10) with $l \neq 0$ and $r \neq 0$ in a similar way to the above case. Therefore in either case there exists $\xi'' \in T_2^*$ such that $\xi''(c)$ takes the form given by (10) with $l \neq 0$ and $r \neq 0$. Put $c_3 = \xi''(c)$. Thus, since for any $\tau \in T_2$ there exists $\beta \in T_2^+$ such that $\tau^{-1} = \beta$, there exists $\xi_3 \in T_2^*$ such that $\xi_3(c_3) = c$. Hence it suffices to show the existence of $\xi_4 \in T_2^+$ such that $\xi_4(c_p) = c_3$ in order to establish the existence of $\xi \in T_2^+$ such that $\xi(c_p) = c$. For we immediately have $\xi_3\xi_4(c_p) = c$, where $\xi_3\xi_4 \in T_2^+$. In the case where c is of the form (10) with $l \neq 0$ and $r \neq 0$, we put $c_3 = c$. We complete the proof by showing the existence of $\xi_4 \in T_2^+$ such that $\xi_4(c_p) = c_3$. Let $\xi_l\zeta_r = 1\ 0\ 0^s\ 1\ 1$. Then we can assume $s = 1$ or 2 because, if $s \geq 3$, then applying $\tau_2\tau_1$ to c_1 decreases s by 2. Let us define $\tau_6, \ldots, \tau_{14} \in T_1$ as follows.

$$\tau_6 = \begin{bmatrix} 0&1&0&0&0&0 \\ 0&1&1&1&1&0 \end{bmatrix}, \quad \tau_7 = \begin{bmatrix} 0&0&0&1&1&1&1&0 \\ 0&1&1&1&0&1&1&0 \end{bmatrix},$$

$$\tau_8 = \begin{bmatrix} 0&0&1&1&1&0 \\ 0&1&1&1&1&0 \end{bmatrix}, \quad \tau_9 = \begin{bmatrix} 0&1&1&1&1&0 \\ 0&1&1&1&0&0 \end{bmatrix},$$

$$\tau_{10} = \begin{bmatrix} 0&1&1&1&1&0&0 \\ 0&1&0&1&1&1&0 \end{bmatrix}, \quad \tau_{11} = \begin{bmatrix} 0&1&1&1&0&0 \\ 0&1&1&1&1&0 \end{bmatrix},$$

$$\tau_{12} = \begin{bmatrix} 0&1&1&1&1&0 \\ 0&0&1&1&1&0 \end{bmatrix}, \quad \tau_{13} = \begin{bmatrix} 0&1&1&1&1&0 \\ 0&1&0&1&1&0 \end{bmatrix},$$

$$\tau_{14} = \begin{bmatrix} 0&1&1&1&1&0&0 \\ 0&1&0&0&1&1&0 \end{bmatrix}.$$

$\xi_4 \in T_2^+$ is composed of the parallel maps $\tau_6, \ldots, \tau_{14}$. ξ_4 transforms c_p to c_1 as follows. To begin with, τ_6 is applied. Then, if $R_0(c_3) > 3$, then $\tau_{11}\tau_{12}$ is applied appropriate times. If $R_0(c_1) < 3$, then $\tau_8\tau_9$ is applied appropriate times. Then, $\tau_8\tau_7$ and $\tau_8\tau_9$ are applied appropriately to construct substring $\zeta_{r-1} \cdots \zeta_1$ from the right to the left. Then $\tau_8\tau_9$ is applied appropriate times. To construct substring $\eta_1 \cdots \eta_{l-1}$ from the left to the right, $\tau_{11}\tau_{10}$ and $\tau_{11}\tau_{12}$ are applied appropriately. Finally, to construct $\eta_l\zeta_r$, τ_{13} is applied if $s = 1$. Otherwise, τ_{14} is applied. Since for any i, $4 \leq i \leq 14$, $\tau_i \in T_1$, the proof is completed.

When $|\Sigma| \geqslant 3$, we can make the neighborhood index simple to obtain the next theorem.

Theorem 19. *If* $|\Sigma| \geqslant 3$, *then* TA $(\Sigma, Z, X_{(-5,5)}, T_{FB})$ *is strongly connective.*

The proof of this theorem is very much like the proof of the last one and is omitted. But we notice that instead of τ_6 we may use $\left[\begin{smallmatrix}0&1&0&0\\0&1&2&0\end{smallmatrix}\right]$, and that, instead of τ_7, we may apply

$$\begin{bmatrix}0 & 0 & 1 & 2 & 0\\0 & a_2 & a_1 & a_1 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0 & a_2 & a_1 & a_1\\0 & a_3 & a_2 & a_2 & a_1 & a_1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0 & 0 & 0 & a_2 & a_1 & a_1\\0 & 1 & 2 & 0 & a_1 & a_1\end{bmatrix},$$

appropriately, where $a_1, a_2, a_3 \in \Sigma - \{0\}$, $a_1 \neq a_2$, and $a_2 \neq a_3$.

References

[1] H. Yamada and S. Amoroso, A completeness problem for pattern generation in tessellation automata, *J. Comput. System Sci.* **4** (1970) 137–176.
[2] A. Maruoka and M. Kimura, Completeness problem of one-dimensional binary scope-3 tessellation automata, *J. Comput. System Sci.* **9** (1974) 31–47.
[3] A. Maruoka and M. Kimura, Decomposition phenomenon in one-dimensional scope-three tessellation automata with arbitrary number of states, *Information and Control* **34** (1977) 296–313.
[4] M. Nasu and N. Honda, A completeness property of one-dimensional tessellation automata, *J. Comput. System Sci.* **12** (1976) 36–48.
[5] S. Amoroso and Y. Patt, Decision procedures for surjectivity and injectivity of parallel maps for tessellation structures, *J. Comput. System Sci.* **6** (1972) 448–464.
[6] D. Richardson, Tessellations with local transformations, *J. Comput. System Sci.* **6** (1972) 373–388.